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INTEGRABILITY OF A NON-AUTONOMOUS COUPLED KdV SYSTEM

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The Painlevé property of coupled, non-autonomous Korteweg-de Vries (KdV) type of systems is studied. The conditions under which the systems pass the Painlevé test for integrability are obtained. For some of the integrable cases, exact solutions are given.

Keywords: non-autonomous KdV systems; Painlevé analysis; exact solutions

1. Introduction

For a better understanding of complicated physical phenomena scientists have experienced that it is necessary to introduce mathematical models whose time evolutions might show some features very similar to those of the original phenomena. These models are usually systems of nonlinear differential equations. These equations can be solved by the use of approximation techniques. But the range of applicability and usefulness of these solutions increase the interest on the exact solutions and on the solution generating methods for nonlinear equations. Before attempting to solve an equation one usually needs to know whether the equation is integrable or not. The difficulty of obtaining solutions by use of inverse scattering transform technique, makes this information valuable. One of the powerful tools to obtain this information is the Painlevé test for integrability. There are strong evidences that all integrable equations have Painlevé property, that is, all solutions are single valued around movable singularities.[1]

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In this work, we consider the non-autonomous, coupled KdV type systems

$$\begin{aligned} u_t &= u_{xxx} + a(t) u u_x, \\ v_t &= v_{xxx} + b(t)(uv)_x, \end{aligned} \tag{1}$$

where $a(t)$ and $b(t)$ are some arbitrary functions. We apply the Painlevé test for integrability to system (1), following the Weiss-Kruskal algorithm of singularity analysis [2,3] and obtain the conditions on the functions $a(t)$ and $b(t)$. We find the subclasses of these equations that possess the Painlevé property. By using the truncated expansions we obtain the exact solutions for some of these subclasses, explicitly.

2. The Non-autonomous Coupled KdV type Systems

Following the approach of Weiss *et al* [2], we assume that the solutions of (1) can be represented by the expansions,

$$\begin{aligned} u(x,t) &= \sum_{r=0}^{\infty} u_r(x,t)\phi^{r+\alpha}, \\ v(x,t) &= \sum_{r=0}^{\infty} v_r(x,t)\phi^{r+\beta}, \end{aligned} \tag{2}$$

where α and β are integers, $u_r(x,t)$ and $v_r(x,t)$ are analytic functions in a neighborhood of the singularity manifold $\phi(x,t)$. A hypersurface $\phi(x,t)=0$ is noncharacteristic of the system (1) if $\phi_t\phi_x \neq 0$. We choose[3] $\phi(x,t) = x + \psi(t)$, without loss of generality, hence the coefficients (u_r, v_r) are independent of x. This is the simplest choice for the test, but it cannot be used to obtain the particular solutions. The substitution of $u(x,t) = \sum_{r=0}^{n} u_r(t)\phi^{r+\alpha}, \quad v(x,t) = \sum_{r=0}^{n} v_r(t)\phi^{r+\beta}$ into (1) determines the branches, i.e. the admissible dominant behavior of solutions, and the corresponding positions r of the resonances where the arbitrary functions can appear in the expansions (2). The leading order analysis gives that

$$\alpha = -2, \quad u_0 = -\frac{12}{a(t)}, \quad (\beta-2)\left[\beta(\beta-1) - 12\frac{b(t)}{a(t)}\right] v_0(t) = 0. \tag{3}$$

The branches satisfying (3) are

$$\beta_1 = 2, \;\; \beta_2 = -m, \;\; \beta_3 = m+1 \tag{4}$$

where m is a non-negative integer and

$$b(t) = \frac{a(t)}{12}(m^2+m). \tag{5}$$

For each branch, the corresponding positions r of resonances are

$$\begin{aligned}\beta_1 &= 2, \;\; r=0, m-1, -(m+2),\\ \beta_2 &= -m, \;\; r=0, m+2, 2m+1,\\ \beta_3 &= m+1, \;\; r=0, 1-m, -(2m+1).\end{aligned} \tag{6}$$

For every branch there exists the common resonance $r=0$. For β_1 and β_3, at least one of the resonances always stands in a negative position. The second branch, $\beta_2=-m$, has two positive resonances for every value of m. Hence, the second branch is generic: the expansions (2) with (3) represent the general solutions near singularity. Next, we find from (1) the recursion relations for the coefficients $u_r(t)$ $(r=0,1,2,\ldots)$ of the expansions (2). We see that the resonances occur at $r=-1,4,6$. The resonance at $r=-1$ corresponds to the arbitrariness of $\phi(x,t)$. For the other resonances $(r=4,6)$ the recursion relations turn out to be consistent if

$$a_{tt}(t)a(t)-3a_t^2(t)=0. \tag{7}$$

On the other hand, the recursion relations for $v_r(t)$ are depend on m. For each value of m, there exist different recursion relations with different resonances, but $r=0$ is common for all those cases. We check every case up to $m=15$ and find the following cases:

$$\begin{aligned}m &= 0, \;\; r=-1,0,1,2,4,6;\\ m &= 2, \;\; r=-1,0,4,4,5,6;\\ m &= 3, \;\; r=-1,0,4,5,6,7;\end{aligned} \tag{8}$$

for which, the compatibility conditions are automatically satisfied for each resonances and the system (1) passes the Painlevé test if $a(t)$ satisfies (7). We find that equation (7) has the solution

$$a(t)=\pm[-2c_1t+2c_2]^{-1/2}, \tag{9}$$

where c_1 and c_2 are integration constants. It follows that the system (1) possesses the Painlevé property if $a(t)=k$ and $a(t)=\frac{k}{\sqrt{t}}$, where k is any non-zero constant.

For $m=3$, it is obvious from (5) that $a(t)=b(t)$ in the system (1). The case $a(t)=b(t)=\frac{k}{\sqrt{t}}$ corresponds to the perturbation system of the cylindrical KdV (cKdV) equation.[4] For the other case, $a(t)=b(t)=k$, the system (1) is the perturbative KdV system and is studied in Ref. 5. In the following sections we study two of the systems (1), corresponding to $m=2$, in detail.

2.1. *Jordan KdV Systems*

We consider the system of equations

$$\begin{aligned}u_t &= u_{xxx}+2kuu_x,\\ v_t &= v_{xxx}+k(uv)_x,\end{aligned} \tag{10}$$

which corresponds to $m = 2$ and passes the Painlevé test. Actually, this system of equations is known as a Jordan KdV system and was studied in Refs. 6 and 7. To gain more information on (10), we define the transformations by truncating the series expansions (2) on constant level as follows:

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \qquad v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2, \tag{11}$$

where $u_2(x,t)$ and $v_2(x,t)$ satisfy equations (10) and can be choosen as $u_2(x,t) = 0$ and $v_2(x,t) = 0$. Inserting above expressions for u and v into system (10) and setting the coefficients of each power of ϕ to zero, we have

$$u_0 = -\frac{6}{k}\phi_x^2, \qquad u_1 = \frac{6}{k}\phi_{xx}, \tag{12}$$

$$\begin{aligned} \phi_t - 4\phi_{xxx} + 3\frac{\phi_{xx}^2}{\phi_x} &= 0, & (13)\\ \phi_{xxxx} - 2\frac{\phi_{xx}\phi_{xxx}}{\phi_x} + \frac{\phi_{xx}^3}{\phi_x^2} &= 0, & (14) \end{aligned}$$

$$\begin{aligned} v_1 = -\frac{v_{0,x}}{\phi_x} &+ \frac{\phi_{xx}}{\phi_x^2}v_0, & (15)\\ v_{0,xx} - 3\frac{\phi_{xx}}{\phi_x}v_{0,x} - 2\left(\frac{\phi_{xxx}}{\phi_x} - 2\frac{\phi_{xx}^2}{\phi_x^2}\right)v_0 &= 0, & (16)\\ v_{0,t} - \left(2\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_{xx}^2}{\phi_x^2}\right)v_{0,x} &= 0. & (17) \end{aligned}$$

By introducing $\psi(x,t)$, such that $\phi_x = \psi^2$, equations (13) and (14) can be written as

$$\begin{aligned} \psi_t - 4\psi_{xxx} &= 0, & (18)\\ \left(\frac{\psi_{xx}}{\psi}\right)_x &= 0. & (19) \end{aligned}$$

These equations have solutions of the form

$$\psi(x,t) = c_1 e^{[\beta(t)+\alpha(t)(x-1)]} + c_2 e^{-[\beta(t)+\alpha(t)(x-1)]}, \tag{20}$$

and

$$\psi(x,t) = c_1 x + c_2 \tag{21}$$

where $\beta(t) = \int \alpha(t)dt$, c_1 and c_2 are constants.

As a special case, we can choose $\alpha(t) = \alpha = \mathit{constant}$ and $c_1 = c_2 = 1$, so that the solution (20) can be written as

$$\psi(x,t) = 2\cosh \alpha\theta, \tag{22}$$

where $\theta(x,t) = x + 4\alpha^2 t$. The corresponding solution for $\phi(x,t)$ is,

$$\phi(x,t) = \frac{1}{\alpha}\sinh 2\alpha\theta + 2\theta + 16\alpha^2 t. \tag{23}$$

Using this solution we see that equations (16) and (17) are satisfied if

$$v_{0,t} - 4\alpha^2 v_{0,x} = 0. \tag{24}$$

Then,

$$v_0(x,t) = f(x + 4\alpha^2 t) = f(\theta), \tag{25}$$

where f is an arbitrary function of its argument. Now, using (12), (15), (23) and (25) in (11) we obtain the exact solutions of Jordan KdV system (10),

$$\begin{aligned} u(x,t) &= -\frac{24}{k}\left[\left(\frac{1+\cosh 2\alpha\theta}{\frac{1}{\alpha}\sinh 2\alpha\theta + 2\theta + 16\alpha^2 t}\right)^2 - \frac{\alpha \sinh 2\alpha\theta}{\frac{1}{\alpha}\sinh 2\alpha\theta + 2\theta + 16\alpha^2 t}\right], \\ v(x,t) &= \frac{f(\theta)}{(\frac{1}{\alpha}\sinh 2\alpha\theta + 2\theta + 16\alpha^2 t)^2} - \frac{\left[\frac{f'(\theta)}{2(1+\cosh 2\alpha\theta)} - \frac{\alpha f(\theta)\sinh 2\alpha\theta}{(1+\cosh 2\alpha\theta)^2}\right]}{(\frac{1}{\alpha}\sinh 2\alpha\theta + 2\theta + 16\alpha^2 t)}. \end{aligned} \tag{26}$$

These functions can be plotted by using $Mathematica$.[8] Some results are given in $Figures$(1) and (2).

Next, we consider the solution in (21) and find the expression for $\phi(x,t)$ as

$$\phi(x,t) = c_1^2\left(\frac{x^3}{3} - 4t\right) + c_1 c_2 x^2 + c_2^2 x \tag{27}$$

that leads to the rational solutions

$$\begin{aligned} u(x,t) &= -\frac{6}{k}\left(\frac{\phi_x^2}{\phi^2} - \frac{\phi_{xx}}{\phi}\right)^2, \\ v(x,t) &= d_1\left(\frac{\phi_x^2}{\phi^2} - \frac{\phi_{xx}}{\phi}\right) + d_2\left(\frac{\phi_x^{3/2}}{\phi^2} - \frac{\phi_{xx}\phi_x^{-1/2}}{2\phi}\right) \end{aligned} \tag{28}$$

of Jordan KdV system (10) where d_1 and d_2 are arbitrary constants.

2.2. *Non-autonomous Jordan KdV Systems*

As a second example, we consider the system of equations

$$\begin{aligned} u_t &= u_{xxx} + \frac{2}{\sqrt{t}}uu_x, \\ v_t &= v_{xxx} + \frac{1}{\sqrt{t}}(uv)_x, \end{aligned} \tag{29}$$

which corresponds to the case $m = 2$ in (8) and passes the Painlevé test. This system of equations is known as non-autonomous Jordan KdV system and is given in Ref. 9. Inserting the expansions

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi}, \qquad v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi}, \tag{30}$$

into (29) and setting the coefficients of each power of ϕ to zero, we obtain

$$u_0 = -6\sqrt{t}\phi_x^2, \qquad u_1 = 6\sqrt{t}\phi_{xx}, \tag{31}$$

$$\phi_t - 4\phi_{xxx} + 3\frac{\phi_{xx}^2}{\phi_x} = 0, \tag{32}$$

$$\phi_{xxxx} - 2\frac{\phi_{xx}\phi_{xxx}}{\phi_x} + \frac{\phi_{xx}^3}{\phi_x^2} + \frac{\phi_x}{6t} = 0, \tag{33}$$

$$v_1 = -\frac{v_{0,x}}{\phi_x} + \frac{\phi_{xx}}{\phi_x^2}v_0, \tag{34}$$

$$v_{0,xx} - 3\frac{\phi_{xx}}{\phi_x}v_{0,x} - 2(\frac{\phi_{xxx}}{\phi_x} - 2\frac{\phi_{xx}^2}{\phi_x^2})v_0 = 0, \tag{35}$$

$$v_{0,t} - (2\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_{xx}^2}{\phi_x^2})v_{0,x} + \frac{5}{6t}v_0 = 0. \tag{36}$$

The equations (32) and (33) are compatible, i.e.$(\phi_{xxxx})_t = (\phi_t)_{xxxx}$, and can be solved by the substitution $\phi_x = \psi^2$, where

$$\begin{aligned} \psi_t - 4\psi_{xxx} &= 0, \\ \psi_{xx} + \left[\frac{x}{12t} + \alpha(t)\right]\psi &= 0. \end{aligned} \tag{37}$$

However, the last equation can only be solved in terms of Airy functions.[10] The result is

$$\psi(x,t) = (1/t)^{1/3}\left[c_1 Ai(z) + c_2 Bi(z)\right] \tag{38}$$

where

$$z(x,t) = \frac{[(-1/t)^{1/3}(x+12c_0)]}{2^{2/3}3^{1/3}}, \quad \alpha(t) = \frac{c_0}{t} \tag{39}$$

and c_0, c_1, c_2 are constants. The corresponding solution for $\phi(x,t)$ is

$$\begin{aligned} \phi(x,t) &= (1/t)^{2/3}(12c_0 + x)[c_1 Ai(z) + c_2 Bi(z)]^2 \\ &+ 2^{2/3}3^{1/3}(1/t)^{1/3}[c_1 Ai'(z) + c_2 Bi'(z)]^2. \end{aligned} \tag{40}$$

Then,

$$\begin{aligned} u &= 6\sqrt{t}(\ln\phi)_{xx} \\ v &= \frac{v_0}{\phi^2} - \frac{1}{\phi}\left(\frac{v_0}{\phi_x}\right)_x, \end{aligned} \tag{41}$$

are the exact solutions of the system of equations (29) if (35) and (36) are satisfied. Note that, these equations are linear in v_0 and a particular solution is $v_0 = C\sqrt{t}\phi_x^2$ with $C = constant$. In this particular case, v is proportional to u, i.e. $v = -(C/6)u$. This implies that the system of equations in (29) reduces to a cKdV equation after

the transformation $u \to \sqrt{t}\, u$.[11–13] In Ref.13, a hierarchy of solutions for the cKdV equation is derived in terms of Airy functions. These solutions can be obtained from (41) together with (40).

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